Modules over a Ringed Space

Def: let $(X,\mathcal{O})$ be a ringed space, a **$\mathcal{O}$-Mdl** M is a sheaf of abelian groups over X such that:

1) $\forall\, U \overset{\circ}{\subset} X$ ($\overset{\circ}{\subset}$ denotes open subset) $M(U)$ is a module over $\mathcal{O}(U)$.

2) for $V \overset{\circ}{\subset} U \overset{\circ}{\subset} X$

$$\begin{tikzcd} M(U) \arrow[r, "e_M"] & M(V) \\ M(U)\times\mathcal{O}(U) \arrow[u, "(-\cdot-)"] \arrow[r, "e_M \times e_{\mathcal{O}}"] & M(V)\times\mathcal{O}(V) \arrow[u, "(-\cdot-)"'] \end{tikzcd}$$

commutes

A **$\mathcal{O}$-Mdl Morphism** $f: M \to N$ is an abelian sheaves morphism s.t. $\forall\, U \overset{\circ}{\subset} X$ $f(U): M(U) \to N(U)$ is a $\mathcal{O}(U)$-module homomorphism.

Ex: 1) For A abelian group denote the constant sheaf $\underline{A}$, then for $(X, \underline{\mathbb{Z}})$ ringed space and X having finitely many connected components M be an abelian sheaf on X, $U \overset{\circ}{\subset} X$, $\lambda \in \underline{\mathbb{Z}}(U)$, $a \in M(U)$ and $\{U_i\}_{i\in I}$ the connected components of U, then $\lambda|_{U_i} = \lambda_i \in \mathbb{Z}$, set $\lambda \cdot a \in M(U)$ as the unique section that restricts to $\{\lambda_i e^{U}_{U_i}(a)\}_{i\in I}$.

So the abelian sheaves correspond to the $\underline{\mathbb{Z}}$-Mdls.

2) M_i $\mathcal{O}$-Mdl $\forall i \in I$, $U \overset{\circ}{\subset} X$, then

$U \mapsto \prod_I M_i(U)$ and $U \mapsto \bigoplus_I M_i(U)$

define $\mathcal{O}$-Mdls (direct product and sum) denoted M^I, $M^{(I)}$

3) $\forall U \overset{\circ}{\subset} X$ and M $\mathcal{O}$-Mdl, we set $M|_U(U') := M(U')$ $\forall U' \overset{\circ}{\subset} U$, then this defines a $\mathcal{O}|_U$-Mdl over the ringed space $(U, \mathcal{O}|_U)$

<u>Def.</u>: L, M $\mathcal{O}$-Mdl, then for the presheaf of $\mathcal{O}(U)$-mdls: $U \mapsto L(U) \otimes_{\mathcal{O}(U)} M(U)$ we define the sheaf $\Gamma L(L \otimes_{\mathcal{O}} M)$ to be the <u>tensor product $\mathcal{O}$-Mdl.</u>

the sheaf of $\mathcal{O}(U)$-mdls: $U \mapsto \hom_{\mathcal{O}|_U}(L|_U, M|_U)$
RHS is the abelian group of $\mathcal{O}|_U$-Mdl homomorphism between $L|_U$ and $M|_U$ is the <u>$\mathcal{O}$-Mdl of homomorphism from L to M.</u>

<u>Rem</u>: 1) for $\mathcal{O}$-Mdl M, $\forall U \overset{\circ}{\subset} X$ $M(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \cong M(U)$
so $\Gamma L(M \otimes_{\mathcal{O}} \mathcal{O}) \cong \Gamma L(M) \cong M$

2) if $\forall U \overset{\circ}{\subset} X$ $L(U), M(U)$ are $\mathcal{O}(U)$-algebras and X is a noetherian top. space then $L \otimes_{\mathcal{O}} M$ is a sheaf since tensor product is the coproduct in the category of algebras so A1 b) (ÜB 5) (Alg Geo 2022)
$\Longrightarrow$ $\Gamma L(L \otimes_{\mathcal{O}} M) = L \otimes_{\mathcal{O}} M$.

From now on we write $L \otimes_{\mathcal{O}} M$ also for its sheafification.

Def: $(X,\mathcal{O})$ ringed space, $\mathcal{M}$ $\mathcal{O}$-Mdl is **locally free of rank $n \in \mathbb{N}_0$** if for $\{U_i\}_{i\in I}$ open cover of X

$$\mathcal{M}|_{U_i} \cong (\mathcal{O}|_{U_i})^n$$

as $\mathcal{O}|_{U_i}$-Mdl $\forall i \in I$

Ex since $(\mathcal{N}|_U)^m = \mathcal{N}^m|_U$, the $\mathcal{O}$-Mdl $\mathcal{O}^m$ is l.f. of rank m.

rem: $\mathcal{M}$ l.f $\mathcal{O}$-Mdl, we see that $\forall x \in X$

$(\mathcal{O}_x)^n \cong \mathcal{M}_x$ as $\mathcal{O}_x$-mdls

so for $(X, \underline{\mathbb{Z}})$, $\mathcal{M}$ $\underline{\mathbb{Z}}$-Mdl and l.f.

$\Rightarrow \mathcal{M}_x \cong \mathbb{Z}^n$ as abelian groups

Ex/Def: X, E top. spaces, $\pi : E \to X$ cont. surj. $k = \mathbb{R}, \mathbb{C}$, then we call (E, π) a **n-vector bundle over k** if

$\forall x \in X$ $\pi^{-1}(x) = k^n$, $n \in \mathbb{N}$ and

$\forall p \in X$ $\exists U \overset{\circ}{\subset} X$ $p \in U$ with

(1) $\qquad \varphi : U \times k^n \longrightarrow \pi^{-1}(U)$ homeo.

and $\pi\varphi(U,v) = U \quad \forall (U,v) \in U \times K^n$

and $\psi : K^n \longrightarrow \pi^{-1}(x)$ linear iso.

$v \longmapsto \varphi(x,v)$

then we define on X the following ab. sheaf:

$$\Gamma E(U) := \{ \sigma : U \to E \mid \sigma \text{ cont. } \pi \circ \sigma = \mathrm{id}_X \}$$

$\forall U \overset{\circ}{\subset} X$ "sheaf of sections of the vector bundle (E,π)"

If we equip X with one of the sheaves $C_{\mathbb{R}}(X)$, $C_{\mathbb{C}}(X)$, and call it $\mathcal{O}$, then:

$$\Gamma E(U) \times \mathcal{O}(U) \longrightarrow \Gamma E(U) \qquad U \overset{\circ}{\subset} X \text{ and}$$
$$(\sigma, f) \longmapsto \sigma \cdot f$$

$$\sigma \cdot f : U \longrightarrow E$$
$$x \longmapsto f(x) \cdot \underbrace{\sigma(x)}_{\in K^n}$$

this makes ΓE into a $\mathcal{O}$-Mod.

For $U \overset{\circ}{\subset} X$ s.t. (E,π) trivial (this means that an homeo as in (1) exists), then $\forall U' \overset{\circ}{\subset} U$:

$$\Gamma E|_U(U') = \{\sigma : U' \to U' \times k^n \mid \sigma \text{ cont}, \pi\sigma = id_{U'}\}$$

$$\Gamma E|_U(U') \xrightarrow{f} \mathcal{O}^n|_U(U')$$

$$\sigma \longmapsto f(\sigma) : U' \to k^n$$
$$x \longmapsto \sigma_2(x)$$

is a group-isomorphism and $\mathcal{O}|_U(U')$-linear

so ΓE is l.f. of rank n.

prop: $(X, \mathcal{O})$ ringed space, then the map

$$\varphi : \mathcal{O}(X) \longrightarrow hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) := End(\mathcal{O})$$

$$s \longmapsto \varphi(s)$$

$$\varphi(s) : \mathcal{O}(U) \longrightarrow \mathcal{O}(U) \qquad \forall U \overset{\circ}{\subset} X$$
$$a \longmapsto e_U^X(s)\, a$$

is a ring isomorphism, for $(End(\mathcal{O}), +, \circ)$ ring with composition of morphism as multiplication.

proof:

i) $\forall s \in \mathcal{O}(X) \quad \forall U' \overset{\circ}{\subset} U \overset{\circ}{\subset} X$

$$e^X_{U'}(s) \cdot e^U_{U'}(x) = e^U_{U'}(e^X_U(s) \cdot x)$$

so $\varphi(s)$ is a morphism of sheaves

and clearly $\mathcal{O}(U)$-linear

so it is a morphism of $\mathcal{O}$-Mdl.

ii) $e^X_U(s \cdot s')(x) = e^X_U(s) \cdot e^X_U(s') \cdot x$

so φ is ring homomorphism.

iii) $\varphi(s) = 0$ then $e^X_X(s) \cdot x = s x = 0$

and for $x = 1 \Rightarrow s = 0$

$\Rightarrow \varphi$ inj.

$f \in End(\mathcal{O})$, then $f_U(a) = f_U(a \cdot 1)$
$= a f_U(1)$

So f_U is given by mult. with

$f_U(1)$, since f is a morph. of s:

$\Rightarrow e(x) \cdot f_U(1) = e(x)\, e(f_X(1))$

$\Rightarrow f_U(1) = e(f_X(1))$

$\Rightarrow \varphi$ surj.

rem: •) $Aut(\mathcal{O}) \cong (\mathcal{O}(U))^{\times}$

•) $Mat_{n\times n}(\mathcal{O}(X)) \cong End_{\mathcal{O}}(\mathcal{O}^n)$

$GL_{n\times n}(\mathcal{O}(X)) \cong Aut_{\mathcal{O}}(\mathcal{O}^n)$

Def: $(X,\mathcal{O})$ ringed space, then we call the locally free $\mathcal{O}$-Mdls of rank 1 the **invertible $\mathcal{O}$-Mdl** and the set of their isomorphism classes **Picard Group** of $(X,\mathcal{O})$

thm $(Pic(X), \otimes)$ is a group

Proof 1) We have already seen that $\mathcal{O}$ is neutral element.
M, N inv. and $\{U_i\}_{i\in I}$ sufficiently fine open cover of X s.t. $M|_{U_i} \cong N|_{U_i} \cong \mathcal{O}|_{U_i}$ $\forall i\in I$, then

$$M \otimes_{\mathcal{O}} N|_U \cong M|_U \otimes_{\mathcal{O}|_U} N|_U \cong \mathcal{O}|_U \otimes_{\mathcal{O}|_U} \mathcal{O}|_U \cong \mathcal{O}|_U$$

$\Rightarrow M \otimes_{\mathcal{O}} N$ is also invertible.

11) We want an iso. of $\mathcal{O}$-Mdls

$$\mathrm{hom}_{\mathcal{O}}(M,\mathcal{O}) \otimes_{\mathcal{O}} M \xrightarrow{\cong} \mathcal{O}$$

we define presheaf morphism:

$$\mathrm{hom}_{\mathcal{O}|_U}(M|_U, \mathcal{O}|_U) \otimes_{\mathcal{O}(U)} M(U) \xrightarrow{\varphi} \mathcal{O}(U)$$

$$f \otimes x \longmapsto f_U(x)$$

this induces: $\tilde{\varphi} : \mathrm{hom}_{\mathcal{O}}(M,\mathcal{O}) \otimes_{\mathcal{O}} M \to \mathcal{O}$

and $\tilde{\varphi}$ is a sheaves-morph.
if we take $U \overset{\circ}{\subset} X$ small enough:

$$\begin{tikzcd}
\mathrm{hom}_{\mathcal{O}|_U}(M|_U, \mathcal{O}|_U) \otimes_{\mathcal{O}(U)} M(U) \arrow[r, "\varphi"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{O}(U) \\
\mathrm{hom}_{\mathcal{O}|_U}(\mathcal{O}|_U, \mathcal{O}|_U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \arrow[d, phantom, "\cong" rotate=90] & \\
\mathcal{O}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U) \arrow[d, phantom, "\cong" rotate=90] & \\
\mathcal{O}(U) \arrow[uuur, "\psi"'] &
\end{tikzcd}$$

and ψ is an iso.

$\Rightarrow \varphi_x$ is an iso $\forall x \in X \Rightarrow \tilde{\varphi}_x$ is an iso

$\Rightarrow \tilde{\varphi}$ is iso of $\mathcal{O}$-Mdls.

Ex: For $(X, C_k(X))$ and N a loc. free $C_k(X) := \mathcal{O}$ - Mdl of rank $n \in \mathbb{N}$.

For $\{U_i\}_{i\in I}$ open cover s.t. $N|_{U_i} \cong (\mathcal{O}|_{U_i})^n$ $\forall i \in I$, we have $\mathcal{O}|_{U_i \cap U_j}$ - Mdl iso :

$$M_{ij} : \mathcal{O}^n|_{U_i\cap U_j} = (\mathcal{O}|_{U_i})^n|_{U_i \cap U_j} \cong N|_{U_i}|_{U_i\cap U_j} = N|_{U_i \cap U_j}$$

$$N|_{U_j}|_{U_i\cap U_j} \cong (\mathcal{O}|_{U_j})^n|_{U_i\cap U_j} = \mathcal{O}^n|_{U_i\cap U_j}$$

hence by prop. $M_{ij} \in GL_n(\mathcal{O}(U_i \cap U_j))$
so we define $\varphi_{ij} : U_i \cap U_j \longrightarrow GL_n(k)$

$$x \longmapsto M^x_{ij}$$

and this is continous.
Set $E_n := \coprod_{i \in I} U_i \times k^n \big/ \sim$

where $y \in U_i$, $y' \in U_j$ and

$(y, v) \sim (y', v') \iff y = y'$ and $M^y_{ij}(v) = v'$

then the canonical proj.

$$\pi_N : E_N \longrightarrow X$$
$$\overline{(y, v)} \longmapsto y$$

defines (E_N, π_N) a n-v.b. over k.

rem: this association is inverse up to isomorphism to the one given at the beginning:

$$\left\{ \begin{matrix} \text{iso class.} \\ \text{of } n\text{-v.b.} \\ \text{over } k \\ \text{on } X \end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix} \text{iso class. of} \\ \text{loc free} \\ C_k(X)\text{-Mdls} \end{matrix} \right\}$$

if we consider tensor product of v.b. on LHS, the latter reduces to a group isomorphism:

$$\left\{ \begin{matrix} \text{iso class.} \\ \text{of 1-dim v.b.} \\ \text{over } k \text{ on } X \end{matrix} \right\} \simeq \operatorname{Pic}(X)$$

hence for the case $(X, C_k(X))$

$\operatorname{Pic}(X) \simeq H^1(X, \mathbb{Z}/_2)$ $\quad k = \mathbb{R}$

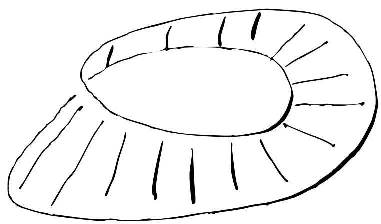
$\operatorname{Pic}(X) \simeq H^2(X, \mathbb{Z})$ $\quad k = \mathbb{C}$

Ex For $(S^1, C_{\mathbb{R}}(S^1))$, $\operatorname{Pic}(S^1) \cong \mathbb{Z}/2\mathbb{Z}$

the nontrivial element is given by

ΓM "sections of Möbius bundle"

and $\Gamma M(S^1) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ cont } f(0) = -f(1) \}$



$= M$

here there are no nonvanishing global sections (intermediate value theorem).

twisting sheaves

take $\mathbb{C}$ with standard topology, set $X_0 = X_1 = \mathbb{C}$ both carrying $\mathcal{O}$, the sheaf of analytic fcts. on $\mathbb{C}$.

$\mathbb{C}^\times =: X_{01} \subseteq X_0$, $\mathbb{C}^\times =: X_{10} \subseteq X_1$, $\varphi_{10} : X_{01} \to X_{10}$, $x \mapsto 1/x$

$\varphi_{10} = \varphi_{01}^{-1}$, $\varphi_{11} = \varphi_{00} = \mathrm{id}_{\mathbb{C}}$, all of which are iso. of ringed spaces

then $X := X_0 \sqcup X_1 /\sim$ for $(i,x) \sim (j,y)$

$\iff$

$\varphi_{ji}(x) = y$

and we obtain a sheaf $\mathcal{O}_X$ that makes $(X, \mathcal{O}_X)$ into a ringed space (rav lecture 8), in particular X can be identified with $\mathbb{P}^1$, the complex projective line, hence we obtained the ringed space $(\mathbb{P}^1, \mathcal{O}_X)$, where $\forall\, W \overset{\circ}{\subset} \mathbb{P}^1$:

$$\mathcal{O}_X(W) = \left\{ (f,g) \,\middle|\, f \in \mathcal{O}(W' \cap X_0),\ g \in \mathcal{O}(W' \cap X_1) \text{ and } f\big|_{W' \cap X_{01}} = g \circ \varphi_{01}\big|_{W' \cap X_{10}} \right\}$$

for $\quad W' = p^{-1}(W) \qquad p : X_0 \sqcup X_1 \longrightarrow \mathbb{P}^1$

we set the following for convenience:

$$X_0 = \left\{ \frac{x_1}{x_0} \,\middle|\, x_1 \in \mathbb{C},\ x_0 \neq 0 \right\} \quad X_1 = \left\{ \frac{x_0}{x_1} \,\middle|\, x_0 \in \mathbb{C},\ x_1 \neq 0 \right\}$$

Now we are ready to define other invertible sheaves on $\mathbb{P}^1$

for $n \in \mathbb{Z}$

$$\mathcal{O}_X(n)(W) := \left\{ (f,g) \,\middle|\, \begin{array}{l} f \in \mathcal{O}(W' \cap X_0) \\ g \in \mathcal{O}(W' \cap X_1) \\ \text{and for } x_0, x_1 \neq 0 \\ f\left(\frac{x_1}{x_0}\right) = \left(\frac{x_1}{x_0}\right)^n g\left(\frac{x_0}{x_1}\right) \end{array} \right\}$$

•) $n=0 \Rightarrow \mathcal{O}_X(0)=\mathcal{O}_X$

•) we have isomorphisms

$$\mathcal{O}_X(n)|_{X_0}(W) \longrightarrow \mathcal{O}_X(W)$$
$$(f,g) \longmapsto f$$

and $\mathcal{O}_X(W) \longrightarrow \mathcal{O}_X(n)|_{X_0}(W)$

$$h\left(\tfrac{x_1}{x_0}\right) \longmapsto \begin{cases} \left(h\left(\tfrac{x_1}{x_0}\right), \left(\tfrac{x_0}{x_1}\right)^n h\left(\tfrac{x_1}{x_0}\right)\right) & x_1 \neq 0 \\ h(0) & x_1 = 0 \end{cases}$$

and analog for X_1, so
so $\mathcal{O}_X(n)$ are locally free $\mathcal{O}_X$-Mdl
of rank 1.

Now we compute global sections of $\mathcal{O}_X(n)$ in order to tell them apart.
$\mathbb{P}^1$ is cpct, connected, so by Liouville $\mathcal{O}_X(\mathbb{P}^1) \cong \mathbb{C}$, so for $n \in \mathbb{N}$ we have a $\mathbb{C}$-Vectorspace isomorphism

$$\mathcal{O}_X(n)(X) \cong \begin{cases} \mathbb{C}[x_0,x_1]_n & n \geq 0 \\ (0) & n < 0 \end{cases}$$

So for $n \geq 0$ $\dim_{\mathbb{C}}(\mathcal{O}_X(n)(X)) = n+1$,

and this shows that $\mathcal{O}_X(n) \not\cong \mathcal{O}_X(m)$

for $n \neq m$ and $n, m \geq 0$.

$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$, so

since $\mathcal{O}_X(n)$ invertible $\forall n \in \mathbb{Z}$

and $\mathcal{O}_X(0) \cong \mathcal{O}_X$, we have that:

$$\mathcal{O}_X(-n) \cong \mathrm{hom}(\mathcal{O}_X(n), \mathcal{O}_X)$$

and this shows tha $\forall n, m$

$\mathcal{O}_X(n) \cong \mathcal{O}_X(m) \Rightarrow n = m$

(by tensoring with big enough $e \in \mathbb{N}$ we alway land in positive degree)

As last remark we see

$$\underbrace{\mathcal{O}_X(1)(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(1)(X)}_{\dim 4} \not\cong \underbrace{\mathcal{O}_X(2)}_{\dim 3}$$

$\Rightarrow$ LHS is not a sheaf.